This article was downloaded by: [Istanbul Technical University]

On: 13 August 2015, At: 08:50 Publisher: Taylor & Francis

Informa Ltd Registered in England and Wales Registered Number: 1072954 Registered office: 5 Howick Place, London, SW1P 1WG



Communications in Statistics - Theory and Methods

Publication details, including instructions for authors and subscription information: <http://www.tandfonline.com/loi/lsta20>

Estimation Based on Generalized Order Statistics from the Burr Model

ZEINHUM F. JAHEEN a

a Department of Mathematics , Assiut University , Egypt Published online: 15 Feb 2007.

To cite this article: ZEINHUM F. JAHEEN (2005) Estimation Based on Generalized Order Statistics from the Burr Model, Communications in Statistics - Theory and Methods, 34:4, 785-794, DOI: 10.1081/ STA-200054408

To link to this article:<http://dx.doi.org/10.1081/STA-200054408>

PLEASE SCROLL DOWN FOR ARTICLE

Taylor & Francis makes every effort to ensure the accuracy of all the information (the "Content") contained in the publications on our platform. However, Taylor & Francis, our agents, and our licensors make no representations or warranties whatsoever as to the accuracy, completeness, or suitability for any purpose of the Content. Any opinions and views expressed in this publication are the opinions and views of the authors, and are not the views of or endorsed by Taylor & Francis. The accuracy of the Content should not be relied upon and should be independently verified with primary sources of information. Taylor and Francis shall not be liable for any losses, actions, claims, proceedings, demands, costs, expenses, damages, and other liabilities whatsoever or howsoever caused arising directly or indirectly in connection with, in relation to or arising out of the use of the Content.

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden. Terms & Conditions of access and use can be found at [http://www.tandfonline.com/page/terms-](http://www.tandfonline.com/page/terms-and-conditions) [and-conditions](http://www.tandfonline.com/page/terms-and-conditions)

Communications in Statistics—'Theory and Methods, 34: 785-794, 2005 Copyright © Taylor & Francis, Inc.

ISSN: 0361-0926 print/1532-415X online DOI: 10.1081/STA-200054408



Order Statistics and Records

Estimation Based on Generalized Order Statistics  
from the Burr Model

Downloaded by [Istanbul Technical University] at 08:50 13 August 2015

ZEINHUM F. JAHEEN

Department of Mathematics, Assiut University, Egypt

The concept of generalized order statistics was introduced by Kamps (1995) to unify several concepts that have been used in statistics such as order statistics, record values, and sequential order statistics. Estimation of the parameters of the Burr type XII distribution are obtained based on generalized order statistics. The maximum likelihood and Bayes methods of estimation are used for this purposes. The Bayes estimates are derived by using the approximation form of Lindley (1980). Estimation based on upper records from the Burr model is obtained and compared by using Monte Carlo simulation study. Our results are specialized to the results of AL-Hussaini and Jaheen (1992) which are based on ordinary order statistics.

Keywords Bayes estimation; Burr model; Generalized order statistics; Monte Carlo simulation; Record values.

Mathematics Subject Classification Primary 62F10; Secondary 62F15.

1. Introduction

As a member of the Burr (1942) family of distributions, which includes 12 types of cumulative distribution functions with a variety of density shapes, the two- parameter Burr type XII (denoted by BurrXII(a, b)) distribution has a probability density function (pdf) of the form

f(x) = abxa-1 (1 + xa)-(b+1), x> 0, (a > 0, b > 0), (1.1)

and a cumulative distribution function (cdf)

F(x) = 1 - (1 + xa)-b, x> 0. (1.2)

Received July 18, 2003; Accepted October 15, 2004

Address correspondence to Zeinhum F. Jaheen, Department of Mathematics, Assiut University, Egypt; E-mail: [zjaheen@hotmail.com](mailto:zjaheen@hotmail.com)

The Burr type XII distribution has been proposed as a lifetime model, and its properties have been studied by Burr and Cislak (1968), Rodriguez (1977), Tadikamalla (1980), and Lewis (1981), among others.

Bayesian inferences based on the Burr type XII distribution and some of its testing measures have been discussed by several authors. Papadopoulos (1978) obtained Bayes estimation for the parameter b and the reliability function when a is known based on Type II censored samples. Evans and Ragab (1983) considered Bayesian estimation based on a discrete prior for the two unknown parameters a and b. AL-Hussaini and Jaheen (1992, 1994) considered estimation of the parameters, reliability, and failure rate functions of the model based on Type-II censored samples from a Bayesian approach. Shah and Gokhale (1993) used the maximum likelihood and maximum product of spacings methods to estimate the parameters of the BurrXII(a, b) distribution and compared the estimates for small and large samples. Ali Mousa (1995) considered empirical Bayes estimation for one of the two shape parameters and the reliability function of the Burr type XII distribution based on Type-II censored data obtained from an accelerated life test. Ali Mousa and Jaheen (2002) considered Bayesian estimation of the parameters of the Burr distribution based on progressively censored samples.

Downloaded by [Istanbul Technical University] at 08:50 13 August 2015

Kamps (1995) introduced the concept of generalized order statistics (gos) as a unified approach to order statistics, record values, and sequential order statistics. The gos are defined using quintile transformation based on the distribution function F.

Let X(1, n, m,k),X(r, n,m,k), k > 1, m is a real number, be gos based on absolutely continuous distribution function F with density function f. The joint density function of the above quantities is given by

/\*X(1,n,m,k),...,X(r,n,m,k)/„ „ \

f (xi,..., xr)

= Cr-^ h[1 - F(x;)]mf(x;))[1 - F(xr)]7r-1f(Xr),

F-1(0+) < Xj <•••< xr < F-1(1), (1.3)

where

yr = k + (n - r)(m + 1) > 0,

Cr-1 = h r = 1 2,..-n 7 n = k. (1.4)

j=1

with n e N, k > 0 and m e R. For more details of gos, see Kamps (1995). In the case m = 0 and k = 1, X(r, n, m, k) reduces to the ordinary r-th order statistics and (1.3) is the joint pdf of r ordinary order statistics, X1:n < X2:n <•••< Xr:n. For various distributional properties of ordinary order statistics, see David (1981) and Arnold et al. (1992). If m = —1 and k = 1, then (1.3) is the joint pdf of r upper record values. For some distributional properties of record values, see Ahsanullah (1995) and Arnold et al. (1998).

Distribution properties of gos from a uniform distribution are given by Ahsanullah (1996). He obtained the minimum variance linear unbiased estimators of the parameters of the two parameters of uniform distribution based on the first m gos Kamps (1996) characterized the uniform distribution based on distribution

Downloaded by [Istanbul Technical University] at 08:50 13 August 2015

properties of subranges of gos. Kamps and Gather (1997) characterized the exponential distributions by distributional properties of gos Cramer and Kamps (1996, 1998, 2001) studied some estimation problems with different sequential k-out-of-n systems. Ahsanullah (2000) gave some distributional properties of the gos from the two parameter exponential distribution. He also obtained the minimum variance linear unbiased estimators of the two parameters and characterized the exponential distribution based on gos. Cramer and Kamps (2000) derived relations for expectations of functions of gos from a class of distributions which includes the exponential, uniform, Pareto, Lomax and, Pearson I. Habibullah and Ahsanullah (2000) obtained estimates for the parameters of the Pareto distribution based on gos. Kamps and Cramer (2001) studied some distribution properties of the gos from the Pareto, power and Weibull distributions. Jaheen (2002) considered the prediction of future gos from a general class of distributions which includes the Weibull, compound Weibull, Burr type XII, Pareto, beta, and Gompertz by using Bayesian two-sample prediction technique.

**2. Maximum Likelihood Estimation**

Suppose that X(1, n, m, k),..., X(r, n, m, k), k> 0 and m e R be a generalized ordered random sample of size r drawn from the BurrXII(a, b) population whose pdf is given by (1.1). The likelihood function (LF) may be obtained from (1.1), (1.2), and (1.3), and written as

*L(a,* b; x) = *Cr—1arbrv(a;* x) *exp[-bH(a;* x)],

where x = *(xx,..., xr),*

v(a; x) = n

i=1

X

a— 1

1 + XV’

r-1

*H(a; x) = (m +* 1) ln(1 + *x“) + yr* ln(1 + *xar),*

i=1

(2.1)

(2.2)

and yr is as given by (1.4).

Assuming that the parameter a is known, the Maximum likelihood (ML) estimate of the parameter b can be shown to be of the form

bML =

*r*

*H(a; x)* ’

(2.3)

where H(a; x) is as given in (2.2). In this case, the Burr type XII distribution is a particular case of the set-up considered in Cramer and Kamps (1996).

When the two parameters a and b are unknown, the likelihood equation for a can be written as

r1

*r*

*a*

+ *^ a, — b[(m + 1)J2x\*mi + yrxarmr] =* 0,

(2.4)

1=1

where, for i = 1, 2r,

, N ln xi

®i = ®(a; xi) = -rr~a. (2-5)

1 + xi

Substituting the value of b given by (2.3) in (2.4), yields a nonlinear equation in a, by solving it numerically we obtain the ML estimate of the parameter a. Then, substitute the ML estimate of a in (2.3), we obtain the ML estimate of b.

3. Bayes Estimation

In the following, Bayesian estimation for the parameters of the Burr type XII distribution is considered for two cases, the first is when the parameter a is known and the second is when the two parameters a and b are assumed to be unknown.

Downloaded by [Istanbul Technical University] at 08:50 13 August 2015

3.1. One Parameter Case (a is Known)

When the parameter a is assumed to be known, we use the gamma conjugate prior density for the parameter b, that was first used by Papadopoulos (1978) and AL-Hussaini et al. (1992), when a was assumed known, in the following form

«a+-

g(b) =r \^b«e-b?, b> 0, (a> 0,0 > 0). (3.1)

t(a + 1)

It follows, from (2.1) and (3.1), that the posterior density of b is then

[P \_l\_ H(a- x)]r+a+-

q-(b I X, a) = [P + (a br+a exp[-b(P + H(a; x)]. (3.2)

T(r + a + -)

Under a squared error loss function, the Bayes estimate of the parameter b is the posterior mean in the form

r + a + - P + H(a; x)

b B =

(3.3)

3.2. Two Parameter Case (a and b are Unknown)

When both of the two parameters a and b are unknown, AL-Hussaini and Jaheen (-992) suggested a bivariate prior density function, given by

g(a b) = gi(b 1 a)g2 (b) (3.4)

where

aa+-

\*-<b 1 a = r(a + l)Pa+-bae-ab"- b> 0, (a> --,P> <»• (15)

is the gamma conjugate prior, that was first used by Papadopoulos (1978), when a was assumed known, and

g2(a) = a > 0 (y > 0,5 > 0), (3.6)

is the gamma density function.

Multiplying g1(b | a) by g2(a), we obtain the bivariate density of a and b, given from (3.4), by

|  |  |  |
| --- | --- | --- |
|  | 1 b |  |
| -a | y + |  |

g(a, b) = A1aa+5ba exp

a > 0, b > 0,

(3.7)

where a > -1, f, y, and 5 are positive real numbers and A-1 = r(£)T(a + 1)y5fa+1.

The four-parameter gamma-gamma prior (3.7) is so chosen that it would be rich enough to cover the prior belief of the experimenter (AL-Hussaini and Jaheen, 1992). It follows, from (2.1) and (3.7), that the joint posterior density function of a and b given the data is thus

Downloaded by [Istanbul Technical University] at 08:50 13 August 2015

where

and

*q2(a, b* | x) = *A2ar+a+5br+ae-a/yv(a;* x)

x exp

b(#(a; x) +

A- 1 = T(r + a + 1)10(x),

*p TO*

I0(x) *= ar+a+5e-a/yv(a;* x)

'0

a > 0, b > 0,

a

H(a; x) + ^

(r+a+1)

*da.*

(3.8)

(3.9)

(3.10)

It is well known that, under squared error loss function, the Bayes estimator of a function, say U(a, b), is the posterior mean of the function and is given by a ratio of two integrals which may be written as

p TO p TO

*E[U(a, b)* | x] *= I* / *U(a, b)q2(a, b* | *x)dadb*

J*0* J*0*

/0°° *10° U(a, b)L(a, b; x)g(a, b)dadb  
f0°* /0°° *L(a, b; x)g(a, b)dadb*

(3.11)

Generally, the ratio of two integrals (3.11) cannot be obtained in a simple closed form. Numerical methods of integration may be used in this case, which can be computationally intensive, especially in high dimensional parameter space. Instead, we can use the approximation form due to Lindley (1980). In the following, a review of Lindley’s approximation form is given.

3.2.1. The Approximation form of Lindley. Lindley (1980) developed approximate procedures for the evaluation of the ratio of two integrals in the form

*fa)(k)emdk/ jg(k)eL(X)dk,* (3.12)

where k = (k1,...,kN), L(k) is the logarithm of the likelihood function, g(A) and m(k) = U(k)g(k) are arbitrary functions of k. From Eq. (3.12), the posterior

expectation of the function U(k), for given x, is

*E[U(k)* | x]

*f U(k)eQ(X)dk  
f eQ^dk*

(3.13)

where Q(k) = L(k) + p(k) is the logarithm of the posterior distribution of k except for the normalizing constant and p(k) = ln g(k). Expanding Q(k) in (3.13) into a Taylor series expansion about the posterior mode of k, Lindley obtained the required expression for E[U(k) | x]. For more details, see Lindley (1980).

In this article, we consider Lindley’s approximation form expanding about the posterior mode. For the two parameter case k = (k1, k2), Lindley’s approximation leads to

Downloaded by [Istanbul Technical University] at 08:50 13 August 2015

*Ub = U(k)* + ^ *[B + Q30B12 + Q21C12 + Q12C21 + Q03B21},* (3.14)

where B = Yi=1Y2=1 Uijiij, Q„i = jglku n,% = °1,2,3, n + % = 3, for i7 j = 1, ^

*d£ U \_ d2U dXP Uij = dkidkj*,

Ui = §, Un = iMj, and for i =j ,

*Bij = (Uiiii + Uj^ij)^ii, Cij = 3Uiiiiiij + Uj^ii'ijj + 2^2j),*

xij is the (i, j)th element in the inverse of the matrix Q\* = (—Q j), i, j = 1, 2 such that Q\* = . Expansion (3.14) is to be evaluated at (k 1,k2), the mode of the

posterior density.

In our case, (kj, k2) = (a, b) and Q is then given by

Q

ln q2 a (r + a + §) ln a + (r + a) ln b

*aa* —+ ln *v(a;* x) — *b H + — ,*

7 " L

(3.15)

where H = H(a; x) is as given in (2.2).

The joint posterior mode, denoted by (a,b), is obtained from (3.15) and is given by

7 *P(r + a) b =* ::—,

*a + fiH*

where a is the solution of the following nonlinear equation

(3.16)

*r + a + §  
a*

1 r

- + J] *Wi —*

7 i=1

*P(r + a)B1a + PH*

= 0,

(3.17)

where

B1 = B1 (a; x)

r— 1

+ *(m + 1)T/xiaai + -Xw*

i=1

and ai is as given by (2.5).

The elements of the inverse of the matrix Q\* = (—Qj), i, j = 1, 2 are given by

a2 (r + a)

tii =

*D*

t12 = t21 =

*a2b2 B1D*

(3.18)

t22 =

b2[r + a + 5 + a2B2]

D :

where

Downloaded by [Istanbul Technical University] at 08:50 13 August 2015

Furthermore,

where

B2 = B2(a; x) = E x>2 + ^

i=1

r

B3 = B3(a; x) = (m + 1^xaffl2 + yrxarm2r,

***i=1***

*D = (r +* a)[r *+* a *+* 5 *+ a2B2] — [abB1* ]2

„ 2(r + a + 5) 2(r + a)

Q12 = 0, Q21 = ~B3, Q30 = 3 B4, Q03 = 71 ,

a3 b3

B4 = B4(a; x) = Exa(1 — xa)®3

i=1

r-1

+ (m + 1)£x?o — xa)®3+yrxa(1 — xa)®3-

i= 1

Substituting the above values in (3.14) yields the Bayes estimate of a function U = U(a, b), of the unknown parameters a and b, given by:

^ W 1

Ub = E[U(a, b) | x] = U + — + — [aWW + bWjU,], (3.19)

where

W = a2[(r + a)Un — b^^ + ^)]

+ b2(r + a + 5 + a2B2 )U22,

W1 = (r + a)[(r + a){2(r + a + 5 — a3B4)}

+ 3a3b2B1B3 — 2abB1 (r + a + 5 + a2B2)], W2 = [2(r + a){(r + a + 5 + a2B2)2

— abB1 (r + a + 5 — a3B4)} + a2bB3 x {(r + a)(r + a + 5 + a2B2) + 2(abB1)2}].

(3.20)

All functions in the right-hand side of (3.19) are to be evaluated at the posterior mode (a, b). Now, the Bayes estimates of a and b are computed as follows:

(i) If U(a, b) = a, we have from (3.19),

(ii)

*a BG = a*

evaluated at the posterior mode (a, lb). If U(a, b) = b, we have from (3.19),

1 +

W1 '

***2D2***

(3.21)

***b BG = b***

1 +

W2 '

2D2

(3.22)

Downloaded by [Istanbul Technical University] at 08:50 13 August 2015

evaluated at the posterior mode (a, b).

**4. Numerical Computations Based on Upper Records**

The upper record values XU(1), XV(2),..., XU^r) of size r can be obtained from the gos scheme as a special case by taking m = —1 and k = 1. In this case, estimation for the parameters of the BurrXII(a, b) distribution based on upper records can be obtained from the above results by taking m = —1, k = 1.

The ML and Bayes estimates of the two unknown parameters a and b are computed and compared based on a Monte Carlo simulation study according to the following steps:

1. For a given vector of prior parameters (a, ft, y, 3), we generate a and b from the joint prior density (3.7). The IMSL (1984) is used in the generation of the gamma random variates.
2. For given a and b obtained in Step 1, we generate n = (5, 8, 12) upper record values from the BurrXII(a, b) with pdf (1.1).
3. The ML estimate of a is computed by solving the nonlinear Eq. (2.4), with m = —1, k = 1, and yr = 1, by using ZSPOW routine from the IMSL (1984) library. Substituting the ML estimate of a we compute the ML estimate of b from b = —r—

MLR ln(1+xa)‘

1. The Bayes estimates of a and b are computed from (3.21) and (3.22) with m = — 1, k = 1 and yr = 1.
2. The squared deviations (^\* — )2 are computed for different sizes n where (\*) stands for an estimate (ML or Bayes) and stands for the parameter (a or b).

Table 1

Estimated Risk (ER) for the ML and Bayes estimates of a and b when (a = 2.5, ft = 1.4, y = 3.4, 3 = 2.0) for different values of n and 1000 repetitions

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| n | ER(a Mlr) | ER(a BR) | ER(b mlr) | ER(b br) |
| 5 | 0.1473 | 0.1435 | 0.2765 | 0.2637 |
| 8 | 0.1395 | 0.1318 | 0.2551 | 0.2449 |
| 12 | 0.1364 | 0.1287 | 0.2482 | 0.2378 |

Table 2

Estimated risk (ER) for the ML and Bayes estimates of a and b when (a = 2.0, = 2.5, y = 2.3, 5 = 3.5) for different values of n and 1000 repetitions

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| n | ER(a MLR) | ER(a br) | ER(b mlr) | ER(b br) |
| 5 | 0.2136 | 0.2033 | 0.3254 | 0.3164 |
| 8 | 0.2087 | 0.1947 | 0.3146 | 0.3055 |
| 12 | 0.1935 | 0.1752 | 0.3027 | 0.2866 |

6. The above steps are repeated 1000 times and the estimated risk (ER) is computed by averaging the squared deviations over the 1000 repetitions. The computational results are displayed in Tables 1 and 2.

5. Concluding Remarks

Downloaded by [Istanbul Technical University] at 08:50 13 August 2015

1. In this article, the ML and Bayes methods of estimation are used for estimation of the parameters of the Burr type XII distribution based on generalized order statistics. Estimation based on upper records from the Burr model are obtained as a special case. It has been noticed, from Tables 1 and 2, that the estimated risks of the estimates decrease as n increases and the Bayes estimates have the smallest estimated risks as compared with their corresponding ML estimates.
2. It has not been possible to provide necessary and sufficient conditions for the existence and uniqueness of the ML estimators for the two unknown parameters of the Burr type XII distribution. A graphical investigation may be used for studied the existence of the global maximum or the local maximum of the likelihood function (2.1) in the case of upper record values.
3. If m = 0 and k = 1 (ordinary order statistics), then yr = n — r + 1 and the Bayes estimators (3.21) and (3.22) reduce to those obtained by AL-Hussaini and Jaheen (1992).
4. If the prior parameters are unknown, the empirical Bayes approach may be used to estimate such parameters, (see, for example, Maritz and Lwin, 1989).

Acknowledgments

The author wishes to thank anonymous referees for their comments which improved

the article.

References

Ahsanullah, M. (1995). Record Statistics. Commack, New York: Nova Science Publishers, Inc.

Ahsanullah, M. (1996). Generalized order statistics from two parameter uniform distribution. Commun. Statist. Theor. Meth. 25(10):2311—2318.

Ahsanullah, M. (2000). Generalized order statistics from exponential distribution. J. Statist. Plann. Infer. 85:85-91.

AL-Hussaini, E. K., Jaheen, Z. F. (1992). Bayesian estimation of the parameters, reliability and failure rate functions of the Burr type XII failure model. J. Statist. Comput. Simul. 41:31-40.

AL-Hussaini, E. K., Jaheen, Z. F. (1994). Approximate Bayes estimators applied to the Burr model. Commun. Statist. Theor. Meth. 23(1):99-121.

AL-Hussaini, E. K., Mousa, M. A. M. A., Jaheen, Z. F. (1992). Estimation under the Burr type XII failure model: a comparative study. Test 1:33-42.

Ali Mousa, M. A. M. (1995). Empirical Bayes estimators for the Burr type XII accelerated life testing model based on type-2 censored data. J. Statist. Comput. Simul. 52:95-103.

Ali Mousa, M. A. M., Jaheen, Z. F. (2002). Statistical inference for the Burr model based on progressively censored data. Comput. Math. Applicat. 43(10-11):1441-1449.

Arnold, B. C., Balakrishnan, N., Nagaraja, H. N. (1992). A First Course In Order Statistics. New York: John Wiley and Sons, Inc.

Arnold, B. C., Balakrishnan, N., Nagaraja, H. N. (1998). Records. New York: John Wiley and Sons, Inc.

Burr, I. W. (1942). Cumulative frequency functions. Ann. Math. Statist. 13:215-232.

Burr I. W., Cislak P. J. (1968). On a general system of distributions: I. Its curve shaped characteristics; II. The sample median. J. Amer. Statist. Assoc. 63:627-635.

Downloaded by [Istanbul Technical University] at 08:50 13 August 2015

Cramer, E., Kamps, U. (1996). Sequential order statistics and k-out-of-n systems with sequentially adjusted failure rates. Ann. Inst. Statist. Math. 48:535-549.

Cramer, E., Kamps, U. (1998). Maximum likelihood estimation with different sequential k-out-of-n system. Adv. Stochastic Models Reliabil. Qual. Safety 7:101-111.

Cramer, E., Kamps, U. (2000). Relations for expectations of functions of generalized order statistics. J. Statis. Plann. Infer. 89:79-89.

Cramer, E., Kamps, U. (2001). Sequential k-out-of-n systems. In: Balakrishnan, N. Rao, C. R. eds. Handbook of Statistics: Advances in Reliability. Vol. 20. ch. 12. Amsterdam: Elsevier, pp. 301-372.

David, H. A. (1981). Order Statistics. 2nd ed. New York: John Wiley and Sons, Inc.

Evans, I. G. Ragab, A. S. (1983). Bayesian inferences given a type 2 censored sample from Burr distribution. Commun. Statist. Theor. Meth. 12:1569-1580.

Habibullah, M., Ahsanullah, M. (2000). Estimation of parameters of a Pareto distribution by generalized order statistics. Commun. Statist. Theor. Meth. 29(7):1597-1609.

IMSL (1984). Reference Manual. Houston, Texas: IMSL, Inc.

Jaheen, Z. F. (2002). On Bayesian prediction of generalized order statistics. J. Statist. Theor. Applicat. 1(3):191—204.

Kamps, U. (1995). A Concept of Generalized Order Statistics. Germany: B. G. Teubner Stuttgart.

Kamps, U. (1996). A characterization of uniform distributions by subranges and its extension to generalized order statistics. Metron 30:37-44.

Kamps, U., Gather, U. (1997). Characteristic properties of generalized order statistics from exponential distributions. Applicationes Mathematicae 24(4):383-391.

Kamps, U., Cramer, E. (2001). On distributions of generalized order statistics. Statistics 35:269-280.

Lewis, A. W. (1981). The Burr distribution as a general parametric family in survivorship and reliability theory applications. Ph.D. Thesis, Department of Biostatistics, University of North Carolina.

Lindley, D. V. (1980). Approximate Bayesian methods. J. Trabajos de Estadistica 31:223-237.

Maritz, J. L., Lwin, T. (1989). Empirical Bayes Methods. 2nd ed. London: Chapman & Hall.

Papadopoulos, A. S. (1978). The Burr distribution as a failure model from a Bayesian approach. IEEE Trans. Rel. R-27(5):369-371.

Rodriguez, R. N. (1977). A guide to the Burr type XII distributions. Biometrika 64(1): 129-134.

Shah, A., Gokhale, D. V. (1993). On maximum product of spacings (MPS) estimation for Burr XII distributions. Commun. Statist. Theor. Meth. 22(3):615-641.

Tadikamalla, P. R. (1980). A look at the Burr and related distributions. Inter. Statist. Rev. 48:337-344.